MODELLING SEASONALITY WITH FRACTIONALLY INTEGRATED PROCESSES

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ABSTRACT

We propose in this article the use of a particular version of the tests of Robinson (1994) for testing seasonally fractionally integrated processes. The tests have standard null and local limit distributions and allow us to test unit and fractional seasonal roots even with different amplitudes at different frequencies. A Monte Carlo experiment is conducted to check the power of the tests against different types of fractional alternatives and, an empirical application, using quarterly data for the U.S. total expenditure of several monetary aggregates is also carried out at the end of the article.

Key words: Seasonal unit roots; Fractional integration; Monte Carlo simulations
JEL classification: C12; C15; C22.

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1. Introduction

Many macroeconomic time series contain important seasonal components and it is a common belief that modellers need to pay specific attention to the nature of seasonality rather than essentially to ignore it. As an alternative to the deterministic approaches based on seasonal dummies, stochastic models based on seasonal differencing have been proposed in recent years. These models implicitly assume that the seasonal component substantially drifts over time and thus, a process observed 's' times per year would be transformed to its 's'-period difference on the assumption that the process contains an integrated seasonal component. Thus, for quarterly data, s = 4,

$$(1 - L^4)\, x_t \;=\; u_t, \qquad t \;=\; 1, 2, \ldots \tag{1}$$

where 'x_t' is the time series we observe and 'u_t' is an I(0) process, defined in this context as a covariance stationary process, with spectral density function that is positive and finite at any frequency. Note that the operator '$(1 – L^4)$' can be factored as '$(1 – L)(1 + L)(1 + L^2)$' and thus, x_t in (1) contains four roots of modulus unity: one at zero frequency; one at two cycles per year, corresponding to frequency π; and two complex pairs at one cycle per year, corresponding to frequencies $\pi/2$ and $3\pi/2$ (of a cycle 2π).

A good deal of empirical work has followed this approach: Hylleberg et al. (1990) found evidence for seasonal unit roots in quarterly U.K. consumption and income, using a procedure that allows tests for unit roots at some seasonal frequencies, without maintaining their presence at all such frequencies. Further evidence in favour of seasonal unit roots was found in Otto and Wirjanto (1990) and Lee and Siklos (1991) for Canadian economic time series; in Hylleberg et al. (1993) for Japanese consumption and income; and also in Linden (1994) for the Finish economy. All these works are based on the Hylleberg et al.'s (1990) procedure. Other seasonal unit root tests are Ghysels et al. (1994), Canova and Hansen (1995), and more recently, Tam and Reimsel (1997), the latter proposing a test for a unit root in the seasonal MA operator.

However, seasonal unit roots are only an extremely specialized form for describing the nonstationary nature of seasonality. Consider, for instance, the process,

$$(1 - L^4)^d \, x_t \;=\; u_t, \qquad t \;=\; 1, 2, \ldots \tag{2}$$

with d > 0 and I(0) u_t. Clearly, x_t has four roots of modulus unity, all with the same integration order d. But (2) can also be extended to present different integration orders for each of the seasonal frequencies, for example,

$$(1 - L^2)^{d_1} \, (1 + L^2)^{d_2} \, x_t \;=\; u_t, \qquad t \;=\; 1, 2, \ldots \tag{3}$$

or more generally,

$$(1 - L)^{d_1} (1 + L)^{d_2} \, (1 + L^2)^{d_3} \, x_t \;=\; u_t, \qquad t \;=\; 1, 2, \ldots, \tag{4}$$

for given real values d_1, d_2 and d_3. Then, x_t will be stationary if all integration orders are smaller than 0.5, and we say that x_t has seasonal long memory at a given frequency if the integration order at that frequency is greater than zero.

Few empirical applications have been carried out in relation to seasonal fractional models. The notion of fractional Gaussian noise with seasonality was initially suggested by Abrahams and Dempster (1979) and Jonas (1981), and extended in a Bayesian framework by Carlin et al. (1985) and Carlin and Dempster (1989). Porter-Hudak (1990) applied a seasonally fractionally integrated model to quarterly U.S. monetary aggregate with the conclusion that a fractional model could be more appropriate than standard ARIMAs. Advantages of seasonally fractionally integrated models for forecasting are illustrated in Ray (1993) and Sutcliffe (1994), and another recent empirical application can be found in Gil-Alana and Robinson (1998).

In the following section we describe several versions of the tests of Robinson (1994) which permit us testing seasonally fractionally integrated processes like (2), (3) and (4). Section 3 contains a Monte Carlo experiment to check the power of the tests against different fractional alternatives. The tests of Robinson (1994) are applied in Section 4 to several U.S. monetary aggregates, while Section 5 contains some concluding comments.

2. The tests of Robinson (1994)

Robinson (1994) proposes LM tests for testing unit roots and other fractionally integrated hypotheses when the roots are located at any frequency on the interval [0, π]. He considers the model

$$y_t \;=\; \beta' z_t \;+\; x_t, \qquad t \;=\; 1, 2, \ldots \tag{5}$$

where y_t is a raw time series; β is a (kx1) vector of unknown parameters; z_t is a (kx1) vector of deterministic regressors, and x_t in (5) satisfies

$$\rho(L;\theta)\, x_t \;=\; u_t, \qquad t \;=\; 1, 2, \ldots \tag{6}$$

where $\rho(L;\theta)$ is a prescribed function of L and the (px1) parameter vector θ, that will adopt different forms depending on the model tested. Thus, for example

$$\rho(L;\theta) \;=\; (1 - L^4)^{d+\theta} \tag{7}$$

when testing (2) for a given real value d; Similarly,

$$\rho(L;\theta) \;=\; (1 - L^2)^{d_1+\theta_1}\,(1 + L^2)^{d_2+\theta_2} \tag{8}$$

when testing (3); or more generally,

$$\rho(L;\theta) \;=\; (1 - L)^{d_1+\theta_1}\,(1 + L)^{d_2+\theta_2}\,(1 + L^2)^{d_3+\theta_3} \tag{9}$$

in case of testing (4) for given real values d_1, d_2 and d_3. Also, u_t in (6) must be an I(0) process, with spectral density

$$f(\lambda;\tau) \;=\; \frac{\sigma^2}{2\pi}\, g(\lambda;\tau) \qquad -\pi \;<\; \lambda \;\leq\; \pi,$$

where the positive scalar σ^2 and the (qx1) vector τ are unknown, but g is of known form. Under the null hypothesis:

$$H_o:\; \theta \;=\; 0, \tag{10}$$

the residuals in (5) and (6) are

$$\hat{u}_t \;=\; \rho(L)\, y_t \;-\; \hat{\beta}' w_t, \qquad t \;=\; 1, 2, \ldots$$

where $\rho(L) = \rho(L;0)$ and $\hat{\beta} \;=\; \left(\sum_{t=1}^{T} w_t\, w_t{}'\right)^{-1} \sum_{t=1}^{T} w_t\; \rho(L)\, y_t; \quad w_t \;=\; \rho(L)\, z_t\,.$

The test statistic, which is derived from the Lagrange Multiplier (LM) principle, is

$$\hat{R} \;=\; \frac{T}{\hat{\sigma}^4}\, \hat{a}'\, \hat{A}^{-1}\, \hat{a} \tag{11}$$

where T is the sample size, and

$$\hat{a} \;=\; \frac{-2\pi}{T} \sum_{j}^{*} \psi(\lambda_j)\, g(\lambda_j; \hat{\tau})^{-1}\, I(\lambda_j)$$

$$\hat{A} \;=\; \frac{2}{T}\left(\sum_{j}^{*} \psi(\lambda_j)\psi(\lambda_j)' \;-\; \sum_{j}^{*} \psi(\lambda_j)\, \hat{\varepsilon}(\lambda_j)' \left(\sum_{j}^{*} \hat{\varepsilon}(\lambda_j)\, \hat{\varepsilon}(\lambda_j)' \right)^{-1} \sum_{j}^{*} \hat{\varepsilon}(\lambda_j)\, \psi(\lambda_j)' \right)$$

$$\psi(\lambda_j) \;=\; \mathrm{Re}\left(\frac{\partial}{\partial \theta} \log \rho(e^{i\lambda_j}; 0) \right); \qquad \hat{\varepsilon}(\lambda_j) \;=\; \frac{\partial}{\partial \tau} \log g(\lambda_j; \hat{\tau});$$

$$\hat{\sigma}^2 \;=\; \frac{2\pi}{T} \sum_{j=1}^{T-1} g(\lambda_j; \hat{\tau})^{-1}\, I(\lambda_j)\,,$$

$I(\lambda_j)$ is the periodogram of $\hat{u}_t$ and $\hat{\tau}$ = arg $\min_{\tau \in T^*} \sigma^2(\tau)$, with T^* as a suitable subset of the R^q Euclidean space. The sum on * is over $\lambda_j = 2\pi j/T$, such that $-\pi < \lambda_j < \pi$. $\lambda_j \notin (\rho_l - \lambda_1, \rho_l + \lambda_1)$, l = 1, 2, …, s such that ρ_l, l = 1, 2, …, s < ∞ are the distinct poles of $\rho(L)$. Note that $\hat{R}$ is a function of the hypothesized differenced series which has short memory under (10), and thus, we must specify the frequencies and integration orders of any seasonal roots.

Robinson (1994) established under regularity conditions that

$$\hat{R} \;\to_d\; \chi_p^2, \qquad as\;\; T \to \infty, \tag{12}$$

and also that the tests are efficient in the Pitman sense, that, under local alternatives of form: H_a: $\theta = \delta T^{-1/2}$, $\hat{R}$ has a $\chi_p^2(\nu)$ distribution, with a non-centrality parameter ν that cannot (when u_t is Gaussian) be exceeded by that of any rival regular statistic. Thus, a test of (10) against the alternative, H_a: $\theta \neq 0$ will reject H_o if $\hat{R} > \chi^2_{p,\alpha}$, where $P(\chi^2 > \chi^2_p) = \alpha.$ Ooms (1997) proposes Wald tests based on Robinson's (1994) model in (5) and (6) but they require efficient estimates of the fractional differencing parameters. He uses a modified periodogram regression estimation procedure due to Hassler (1994). Also, Hosoya (1997) establishes the limit theory for long memory processes with the singularities not restricted at the zero frequency and proposes a set of quasi log-likelihood statistics to be applied in raw time seires. Unlike these methods, Robinson's (1994) tests do not require estimation of the long memory parameters since the differenced series have short memory under the null. An empirical application of the tests of Robinson (1994) with $\rho(L) = (1 - L)^d$, i.e., imposing the root exclusively at the zero frequency, (but not at the seasonal ones), can be found in Gil-Alana and Robinson (1997), and given the recent extensive theoretical literature based on seasonal fractional integration, a further study of Robinson's (1994) tests in this context seems overdue.

3. A Monte Carlo experiment

We examine in this section the finite-sample behaviour of versions of the above tests by means of Monte Carlo simulations. In Robinson (1994) a finite-sample experiment was also conducted, looking at the rejection frequencies of the tests when the true model was a random walk, (i.e., $(1 - L)\, x_t = \varepsilon_t$), and the alternatives were either fractional, (i.e., $(1 - L)^{1+\theta}\, x_t = \varepsilon_t$), or autoregressive, (i.e., $(1 - (1 + \theta L))\, x_t = \varepsilon_t$), for different values of θ. A similar study was also carried out by Gil-Alana (1999) for monthly data.

We investigate here the power of the tests of Robinson (1994) when the true model contains four unit roots, that is, (1) with white noise u_t, and the alternatives are seasonally fractionally integrated, first, with the same integration order at all frequencies, i.e., (2), and then allowing different orders of integration at each of the seasonal frequencies, i.e., (3) and (4) for different real values d, d_1, d_2 and d_3.

Across Tables 1-3 we look at the rejection frequencies of Robinson's (1994) tests in (5) and (6) with $\beta = 0$ (i.e., $y_t = x_t$), the true model including (7) with d = 1 and $\theta = 0$. The alternatives will be in all cases fractional, with $\rho(L; \theta)$ given by (7), (8) and (9) with d, d_1, d_2 and d_3, equal to 1 and values of θ, θ_1, θ_2 and θ_3 equal to –1, -0.75, …, (0.25), …, 0.75 and 1. Thus, the rejection frequencies corresponding to $\theta = 0$ will indicate the sizes of the tests. We generate Gaussian series generated by the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Vetterling (1986), with 10,000 replications of each case. The sample sizes are T = 120, 240 and 360 observations and in all cases the nominal size is 5%.

Each table shows in the upper part the rejection frequencies when $\rho(L; \theta) = (1 - L^4)^{1+\theta}$, that is, imposing the same integration order at each frequency. Then, we take $\rho(L; \theta) = (1 - L^2)^{1+\theta_1} (1 + L^2)^{1+\theta_2}$, i.e., allowing different orders of integration for the real and the complex roots, and finally, in the lower part of the tables, we take $\rho(L; \theta) = (1 - L)^{1+\theta_1} (1 + L)^{1+\theta_2}(1 + L^2)^{1+\theta_3}$, i.e., allowing different integration orders at each frequency.

Starting with $\rho(L; \theta) = (1 - L^4)^{1+\theta}$, we observe that the size of $\hat{R}$ is too large in all cases though it approximates to the nominal value with T. Thus, it is 17.8% when T = 120; it becomes 10.1% when T = 240, and reduces to 8.0% with T = 360. We also observe that the test statistic is slightly biased toward positive values of θ, obtaining higher rejection frequencies for $\theta > 0$ than for $\theta < 0$, and this is observed even when T =

360, (Table 3), though in this case, the rejection frequencies are relatively high even for $\theta = -1$, (0.831).

(Tables 1 – 3 about here)

Taking $\rho(L; \theta) = (1 - L^2)^{1+\theta_1} (1 + L^2)^{1+\theta_2}$, the sizes are now 6.9% with T = 120; 6.2% with T = 240, and 5.3% with T = 360. The rejection frequencies are relatively high in all cases, though a bias in favour of positive values of θ_1 and θ_2 is again observed. We see that the lowest value (apart from that corresponding to the true model) is obtained in all cases when $\theta_1 = \theta_2 = -0.25$, in which case the rejection frequencies are 0.160 with T = 120; 0.502 with T = 240 and 0.793 with T = 360.

The rejection frequencies of the tests of Robinson (1994) with $\rho(L; \theta) = (1 - L)^{1+\theta_1} (1 + L)^{1+\theta_2}(1 + L^2)^{1+\theta_3}$ are given in the lower part of Tables 1 – 3. A bias also appears in these cases though it tends to disappear with T. Thus, for example, if $\theta_1 = \theta_2 = \theta_3 = -0.50$, the rejection frequency with T = 120 is 0.580, while the value corresponding to the alternative $\theta_1 = \theta_2 = \theta_3 = 0.50$ for the same T is 0.921. Moreover, increasing the sample size, these values are 0.905 and 1 with T = 240, and 0.971 and 1 with T = 360. Finally, imposing different θ_i's, we also see that the rejection frequencies increase with T, obtaining values higher than 0.97 in all cases when T = 360.

The same experiment was also conducted imposing the true model to be (2) with d = 0.50, 0.75, 1.25 and 1.50, obtaining results, in terms of size and power, similar to those reported across Tables 1 – 3. Therefore, we can conclude this section by saying that the different versions of the tests of Robinson (1994) analysed in this article seem to perform quite well when testing the null hypothesis of four seasonal unit or fractional roots. Clearly, when testing (10) against (7) the tests have greater power than when directed against (8) or (9), though the latter forms will have power against a wider range of alternatives. Finally, the performance of the tests was also evaluated against alternatives of form: $(1 - \rho L^4)x_t = \varepsilon_t$, with different values of ρ, comparing the results of Robinson's (1994) tests with those based on standard seasonal unit root tests (i.e., Dickey, Hasza and Fuller, DHF, 1984). As we should expect, DHF (1984) perform better against these AR alternatives, but worse against the fractional alternatives entertained in this article.

4. An empirical application

Robinson (1994) tests are applied in this section to several U.S. monetary aggregates. The series are the (seasonally unadjusted) total expenditure for M1, M2, M3 and L in the U.S. from 1960.1 to 1998.4 quarterly. Denoting any of the series y_t, we employ throughout the null models

$$y_t \;=\; \beta_1 \;+\; \beta_2\, t \;+\; x_t, \qquad t \;=\; 1, 2, ... \tag{13}$$

$$\rho(L)\, x_t \;=\; u_t, \qquad t \;=\; 1, 2, ... \tag{14}$$

with $\rho(L) = (1 - L^4)^d$; $(1 - L^2)^{d_1}(1 + L^2)^{d_2}$; and $(1 - L)^{d_1}(1 + L)^{d_2}(1 + L^2)^{d_3}$, and d, d_1, d_2, d_3 equal to 0, 0.25, …, (0.25),…, 1.75 and 2. We model the I(0) disturbances u_t as white noise and treat separately the cases $\beta_1 = \beta_2 = 0$ a priori, (i.e., including no regressors in the undifferenced regression); β_1 unknown and $\beta_2 = 0$ a priori, (i.e., including an intercept); and β_1 and β_2 unknown, (i.e., with a linear time trend). However, since the results were very similar across all these cases, we only report here those corresponding to the case of $\beta_1 = \beta_2 = 0$ a priori, i.e., with $y_t = x_t$. (The results were either unaffected by the inclusion of seasonal dummy variables in the regression model (13)).

Table 4 shows the results of the statistic $\hat{R}$ in (11), firstly with $\rho(L) = (1 - L^4)^d$, i.e., imposing the same integration order at each frequency. We see that the results are very similar for the different monetary aggregates, and the non-rejection values occur in all cases when d = 1, 1.25 and 1.50. We also observe that when d = 2, the null is less strongly rejected than for example when d = 0, but on the whole, these extreme values are always rejected, suggesting that the optimal local power properties of Robinson’s (1994) tests may be supported by reasonable performance against non-local alternatives. The lowest statistics across the different values of d are obtained when d = 1.25 for M1, and when d = 1 for the remaining aggregates.

(Table 4 about here)

In view of the similarities observed across the different monetary aggregates, we concentrate now ahead only on M2 as the series of interest, and look at $\hat{R}$ in (11) with $\rho(L) = (1 - L^2)^{d_1} (1 + L^2)^{d_2}$ and $(1 - L)^{d_1} (1 + L)^{d_2}(1 + L^2)^{d_3}$. Allowing a different integration order at the real and at the complex roots, (i.e., with $\rho(L) = (1 - L^2)^{d_1} (1 + L^2)^{d_2}$), we observe very few non-rejection values, all them occurring when d_1 ranges between 1 and 2, and when d_2 ranges between 0 and 0.75. Thus, we observe higher integration orders at the 0 and π frequencies than over the complex ones $\pi/2$ and $3\pi/2$. In fact, H_o (10) is now decisively rejected when $d_1 = d_2 = 1$, 1.25 and 1.50, contradicting the results obtained just above, and the lowest statistic is obtained when $d_1 = 1.50$ and $d_2 = 0.25$, suggesting the importance of the real roots over the complex ones. This contradictory results may be related with the different sizes and rejection frequencies observed in Table 1. If the true model were given by $(1 - L^4)x_t = u_t$, performing the tests of Robinson (1994) with (3) and $d_1 = 1.50$ and $d_2 = 0.25$, we saw in Table 1 that the rejection frequency was exactly 1. On the contrary (and though it is not reported here), performing the opposite experiment, i.e., testing (2) with d = 1 when the true model is given by $(1 - L^2)^{1.50} (1 + L^2)^{0.25}x_t = u_t$, the rejection frequency was 0.144, suggesting both experiments that a model with different orders of integration might be more appropriate for this series. Extending the model, and thus allowing a different integration order at each frequency, we only observe two non-rejection values, corresponding to $d_1 = 1$ and $d_2 = d_3 = 0$, (i.e., a random walk), and $d_1 = 1.50$; $d_2 = 0.50$ and $d_3 = 0.25$. These two possibilities were not allowed with the previous specifications for $\rho(L)$. The results here emphasize once more the importance of the real roots, in particular, the one at the zero frequency.

The non-rejection values obtained across Table 4 are all based on the asymptotic critical values given by the chi-squared distributions. However, since the sample size in this empirical application is not very large, we have also calculated the finite sample critical values of the tests by means of Monte Carlo simulations. The results for the three functional forms of $\rho(L)$ with T = 120 are given in Table 5. We see that, in all cases, the critical values are slightly greater than those given by the chi-squared distributions. Thus, some of the values of $\hat{R}$ in Table 4 where H_o (10) was not rejected when using the asymptotic critical values might now be rejected with the finite sample ones.

(Table 5 about here)

Table 6 is analogous to Table 1, showing the rejection frequencies of Robinson's (1994) tests when T = 120, for the three functional forms of $\rho(L)$, but using now the finite sample critical values obtained in Table 5. We see that taking $\rho(L) = (1 - L^4)^{1+\theta}$, the size of the test reduces considerably (from 17.8% in Table 1 to 4.6% in Table 6). This small size is also associated with some inferior rejection frequencies compared with Table 1, being particularly worrisome the low value obtained with θ = -0.25 (0.077). Imposing $\rho(L) = (1 - L^2)^{d_1} (1 + L^2)^{d_2}$ and $(1 - L)^{d_1} (1 + L)^{d_2}(1 + L^2)^{d_3}$, the sizes also reduce, getting closer to the nominal value of 5%, (4.8% and 4.9% respectively), but the rejection frequencies keep relatively high in practically all cases.

(Table 6 about here)

Looking again at the results in Table 4 we see that when using the finite sample critical values, the proportion of non-rejection values is higher. Thus, if $\rho(L) = (1 - L^4)^d$, H_o (10) is not rejected if d = 1, 1.25, 1.50 and 1.75 (and if d = 2 for M1). This is something to be expected in view of the lower rejection frequencies observed in Table 6.

If $\rho(L) = (1 - L^2)^{d_1} (1 + L^2)^{d_2}$, we observe just one extra non-rejection value corresponding to $d_1 = 1.75$ and $d_2 = 0.75$. Finally, if $\rho(L) = (1 - L)^{d_1} (1 + L)^{d_2}(1 + L^2)^{d_3}$, the non-rejection values are exactly the same as when using the asymptotic critical values, i.e. $d_1 = 1$; $d_2 = d_3 = 0$, and $d_1 = 1.50$; $d_2 = 0.50$; $d_3 = 0.25$. Thus, we may conclude by saying that there is not a large difference in the results whether we use the asymptotic or the finite sample critical values.

The results reported in this article indicate that the orders of integration may differ across the frequencies in spite of the non-rejection values obtained when testing when the same integration order at each frequency is imposed. As a final remark, we are concerned with the possibility that the true model has a different integration order at each frequency, (as it may be the case in this empirical application), but we test imposing the same order of integration at all frequencies. We see, in the lower part of Table 4, that there are two non-rejected models for M2, one which is a random walk, i.e.,

$$(1 - L)\, x_t \;=\; u_t, \qquad t \;=\; 1, 2, ... \tag{15}$$

and the other,

$$(1 - L)^{1.50} (1 + L)^{0.50} (1 + L^2)^{0.25}\, x_t \;=\; u_t, \qquad t \;=\; 1, 2, ..., \tag{16}$$

both with white noise u_t. It was shown in Gil-Alana (1999) that if the true model is given by (15) and we apply Robinson's (1994) tests with $\rho(L) = (1 - L^4)$, the rejection frequency with T = 120 was 0.074. Similarly, we have performed the same experiment with the true model given by (16), obtaining a rejection frequency of 0.099. These extremely low values may be the reason why H_o (10) is not rejected in some cases when testing with $\rho(L) = (1 - L^4)^d$ in the upper part in Table 4. Extending the analysis, and testing (2) when the true model has different orders of integration at each frequency, the rejection frequencies were very low in all cases, suggesting the low power of the tests when we impose the same order of integration at all frequencies.

5. Concluding comments

We have presented in this article different versions of the tests of Robinson (1994) for testing seasonally fractionally integrated processes. The tests have several distinguishing features which make them particularly useful in the applied work: they have standard null and local limit distributions, and this limit behaviour holds across the different hypothesized values of d; Also, they allow us to test different orders of integration at different frequencies and, unlike other procedures, do not require estimation of the fractional differencing parameters.

A Monte Carlo experiment was conducted to check the power of the tests against different fractional alternatives. The results suggest that the tests of Robinson (1994) perform quite well for testing seasonal unit or fractional roots when the same order of integration is imposed at all frequencies. However, if the true model contains different integration orders for the different frequencies, the tests may have very low power, especially if the sample size is not very large.

The tests were also applied to the total expenditure of several monetary aggregates of the U.S. with quarterly data, obtaining results which indicate the importance of the root at the zero frequency over the others. Thus, even if the tests cannot reject a null of four unit roots, the results may be hiding the importance of some of the roots over the others, in particular, the one corresponding to the zero frequency.

The article can be extended in several directions. The Monte Carlo simulations can be extended to study the power of the tests when the disturbances are weakly parametrically autocorrelated, and this can also be done in the empirical application carried out in Section 4. Then, a model selection criterion should be established to determine which might be the correct model specification for these and other macroeconomic time series.

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TABLE 1

Rejection frequencies of $\hat{R}$ in (11) against fractional alternatives.
True model: $(1 - L^4)\, y_t = \varepsilon_t$, and $T = 120$.

$$\rho(L;\theta) = (1-L^4)^{1+\theta}$$

θ	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
	0.604	0.566	0.447	0.153	**0.178**	0.763	0.989	1.000	1.000

$$\rho(L;\theta) = (1-L^2)^{1+\theta_1}\,(1+L^2)^{1+\theta_2}$$

θ_1 / θ_2	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
-1.00	0.790	0.959	0.841	0.833	0.881	0.915	0.939	0.954	0.963
-0.75	0.973	0.695	0.803	0.811	0.921	0.962	0.981	0.988	0.993
-0.50	1.000	0.981	0.493	0.648	0.913	0.979	0.995	0.998	0.999
-0.25	1.000	1.000	0.974	0.160	0.709	0.969	0.996	0.999	0.999
0.00	1.000	1.000	0.999	0.951	**0.069**	0.795	0.988	0.998	0.999
0.25	1.000	1.000	1.000	0.999	0.909	0.402	0.845	0.993	0.999
0.50	1.000	1.000	1.000	1.000	0.999	0.890	0.888	0.877	0.995
0.75	1.000	1.000	1.000	1.000	1.000	0.998	0.898	0.992	0.899
1.00	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.909	0.999

$$\rho(L;\theta) = (1-L)^{1+\theta_1}\,(1+L)^{1+\theta_2}\,(1+L^2)^{1+\theta_3}$$

(θ_1, θ_2) / θ_3	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
(-1.00, -1.00)	0.848	0.930	0.999	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, -0.50)	0.989	0.988	0.999	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.00)	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.50)	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 1.00)	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -1.00)	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -0.50)	1.000	0.962	0.580	0.834	0.998	1.000	1.000	1.000	1.000
(-0.50, 0.00)	1.000	0.998	0.968	0.990	1.000	1.000	1.000	1.000	1.000
(-0.50, 0.50)	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, 0.00)	1.000	1.000	0.998	0.582	**0.073**	0.756	0.999	1.000	1.000
(0.00, 0.50)	1.000	1.000	1.000	0.931	0.733	0.971	1.000	1.000	1.000
(0.00, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, 0.00)	1.000	1.000	1.000	1.000	0.998	0.998	1.000	1.000	1.000
(0.50, 0.50)	1.000	1.000	1.000	1.000	0.799	0.497	0.921	1.000	1.000
(0.50, 1.00)	1.000	1.000	1.000	1.000	0.918	0.642	0.947	1.000	1.000
(1.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.50)	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000
(1.00, 1.00)	1.000	1.000	1.000	1.000	1.000	0.985	0.983	1.000	1.000

10,000 replications were used for each case. Sizes are in bold and the nominal size was 5%.

TABLE 2									
Rejection frequencies of $\hat{R}$ in (11) against fractional alternatives. True model: $(1 - L^4)\, y_t = \varepsilon_t$, and $T = 240$.									
$\rho(L;\theta) = (1-L^4)^{1+\theta}$									
θ	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
	0.750	0.781	0.814	0.554	**0.101**	0.958	1.000	1.000	1.000
$\rho(L;\theta) = (1-L^2)^{1+\theta_1}(1+L^2)^{1+\theta_2}$									
θ_1 / θ_2	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
-1.00	0.910	0.993	0.922	0.950	0.969	0.982	0.989	0.993	0.995
-0.75	0.998	0.891	0.919	0.972	0.992	0.998	0.999	0.999	0.999
-0.50	1.000	1.000	0.846	0.961	0.998	1.000	1.000	1.000	1.000
-0.25	1.000	1.000	0.999	0.502	0.990	0.999	1.000	1.000	1.000
0.00	1.000	1.000	1.000	0.999	**0.062**	0.995	1.000	1.000	1.000
0.25	1.000	1.000	1.000	1.000	0.999	0.762	0.997	1.000	1.000
0.50	1.000	1.000	1.000	1.000	1.000	0.997	0.999	0.998	1.000
0.75	1.000	1.000	1.000	1.000	1.000	1.000	0.997	1.000	0.997
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	1.000
$\rho(L;\theta) = (1-L)^{1+\theta_1}(1+L)^{1+\theta_2}(1+L^2)^{1+\theta_3}$									
(θ_1, θ_2) / θ_3	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
(-1.00, -1.00)	0.963	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -0.50)	1.000	0.998	0.905	0.988	1.000	1.000	1.000	1.000	1.000
(-0.50, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, 0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, 0.00)	1.000	1.000	1.000	0.926	**0.066**	0.972	1.000	1.000	1.000
(0.00, 0.50)	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000
(0.00, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, 0.50)	1.000	1.000	1.000	1.000	1.000	0.965	1.000	1.000	1.000
(0.50, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

10,000 replications were used for each case. Sizes are in bold and the nominal size was 5%.

TABLE 3									
Rejection frequencies of $\hat{R}$ in (11) against fractional alternatives. True model: $(1 – L^4)\, y_t = \varepsilon_t$, and T = 360.									
$\rho(L;\theta) = (1-L^4)^{1+\theta}$									
θ	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
	0.831	0.861	0.927	0.847	**0.080**	0.993	1.000	1.000	1.000
$\rho(L;\theta) = (1-L^2)^{1+\theta_1}(1+L^2)^{1+\theta_2}$									
θ_1 / θ_2	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
-1.00	0.944	0.994	0.958	0.977	0.988	0.993	0.995	0.996	0.998
-0.75	1.000	0.937	0.957	0.991	0.998	0.999	1.000	1.000	1.000
-0.50	1.000	1.000	0.946	0.994	0.999	1.000	1.000	1.000	1.000
-0.25	1.000	1.000	1.000	0.793	0.999	1.000	1.000	1.000	1.000
0.00	1.000	1.000	1.000	1.000	**0.053**	0.999	1.000	1.000	1.000
0.25	1.000	1.000	1.000	1.000	1.000	0.936	1.000	1.000	1.000
0.50	1.000	1.000	1.000	1.000	1.000	0.936	1.000	1.000	1.000
0.75	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\rho(L;\theta) = (1-L)^{1+\theta_1}(1+L)^{1+\theta_2}(1+L^2)^{1+\theta_3}$									
(θ_1, θ_2) / θ_3	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
(-1.00, -1.00)	0.979	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -0.50)	1.000	1.000	0.971	0.999	1.000	1.000	1.000	1.000	1.000
(-0.50, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, 0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, 0.00)	1.000	1.000	1.000	0.996	**0.056**	1.000	1.000	1.000	1.000
(0.00, 0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, 0.50)	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000
(0.50, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

10,000 replications were used for each case. Sizes are in bold and the nominal size was 5%.

TABLE 4									
Testing (10) in (6) with white noise u_t									
$\rho(L;\theta) = (1-L^4)^d$									
Series / d	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
M1	370.7	340.7	162.6	24.30	**2.34'**	**.0009'**	**1.22'**	4.06'	5.94'
M2	368.5	270.1	76.81	7.54	**0.09'**	**0.85'**	**2.89'**	4.96'	6.72
M3	369.9	265.4	71.54	6.36	**0.03'**	**0.90'**	**2.78'**	4.66'	6.27
L	372.5	265.2	70.22	6.51	**0.06'**	**0.81'**	**2.66'**	4.54'	6.46
SERIES: M2	$\rho(L;\theta) = (1-L^2)^{d_1} (1+L^2)^{d_2}$								
d_1 / d_2	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.00	652.2	656.8	657.8	657.0	655.5	653.7	651.8	649.9	648.1
0.25	474.1	505.8	517.1	518.2	514.6	508.9	502.5	495.9	489.5
0.50	130.2	17.84	202.4	209.3	207.1	200.7	192.9	184.9	177.3
0.75	17.53	41.25	59.51	67.35	67.69	64.33	59.66	54.86	50.40
1.00	**1.40'**	11.78	26.15	35.30	37.80	36.23	32.99	29.41	26.05
1.25	**0.71'**	**2.27'**	12.51	22.95	28.01	28.28	26.12	23.17	20.22
1.50	**4.57'**	**0.01'**	**4.41'**	14.51	22.24	24.92	24.16	21.89	19.26
1.75	10.50	**2.72'**	**0.33'**	7.14'	16.58	22.19	23.46	22.25	20.09
2.00	17.57	9.53	**1.27'**	**1.50'**	9.89	18.17	22.18	22.66	21.34
SERIES: M2	$\rho(L;\theta) = (1-L)^{d_1} (1+L)^{d_2} (1+L^2)^{d_3}$								
(d_1, d_2) / d_3	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
(0.00, 0.00)	1004.5	1012.9	1018.3	1022.0	1024.7	1026.7	1028.0	1028.8	1029.3
(0.00, 0.50)	1013.4	1019.9	1023.9	1026.7	1028.6	1029.8	1030.7	1031.1	1031.3
(0.00, 1.00)	1017.8	1023.1	1026.4	1028.2	1029.6	1030.4	1030.9	1031.0	1031.0
(0.00, 1.50)	1020.6	1024.9	1027.3	1028.8	1029.7	1030.2	1030.4	1030.4	1030.2
(0.00, 2.00)	1022.5	1025.9	1027.8	1028.9	1029.5	1029.8	1029.8	1029.6	1029.3
(0.50, 0.00)	165.25	224.7	270.11	306.8	338.0	365.2	389.3	410.9	430.6
(0.50, 0.50)	240.9	306.6	354.7	392.8	424.7	452.4	476.8	498.6	518.3
(0.50, 1.00)	287.4	348.5	392.3	426.8	455.8	481.8	503.5	523.7	542.1
(0.50, 1.50)	321.9	376.8	416.1	447.3	473.7	496.9	517.7	536.6	553.9
(0.50, 2.00)	349.9	398.9	434.4	462.8	487.1	508.7	528.3	546.2	562.7
(1.00, 0.00)	**1.82'**	9.43	19.48	27.02	32.49	36.70	40.10	42.92	45.32
(1.00, 0.50)	11.93	25.24	41.65	55.52	67.13	77.24	86.29	94.52	102.0
(1.00, 1.00)	23.08	43.94	64.56	81.40	95.45	107.7	118.7	128.7	138.0
(1.00, 1.50)	34.06	58.86	80.11	96.78	110.4	122.3	133.0	142.7	151.8
(1.00, 2.00)	45.09	71.09	91.48	107.1	119.8	130.8	140.8	150.0	158.6
(1.50, 0.00)	13.05	22.02	37.02	49.67	59.49	67.35	73.84	79.35	84.13
(1.50, 0.50)	13.44	**6.36'**	12.19	19.14	24.62	28.82	32.11	34.72	36.81
(1.50, 1.00)	15.53	11.10	19.73	29.37	37.49	44.30	50.20	55.39	60.03
(1.50, 1.50)	15.46	17.81	30.20	42.06	51.82	60.05	67.25	73.70	79.57
(1.50, 2.00)	16.17	24.75	39.21	51.59	61.42	69.56	76.65	83.01	88.83
(2.00, 0.00)	25.69	43.36	65.23	82.72	95.77	105.7	113.7	120.3	125.9
(2.00, 0.50)	25.17	13.30	18.83	28.03	36.32	43.21	48.99	53.89	58.10
(2.00, 1.00)	31.93	11.15	10.28	15.39	20.67	25.05	28.61	31.49	33.80
(2.00, 1.50)	31.17	12.13	13.38	20.35	27.29	33.23	38.30	42.68	46.49
(2.00, 2.00)	26.64	13.53	18.82	28.04	36.36	43.32	49.25	54.42	59.02

In bold the non-rejection values of the null hypothesis (10) at the 95% significance level. ': Non-rejection values at the 95% significance level with the finite sample critical values obtained in Table 5

TABLE 5				
Critical values of Robinson’s (1994) tests in finite samples				
ρ(L) / Size	10%	5%	1%	0.1%
$\rho(L) = (1 - L^4)^d$	5.00	6.32	8.88	12.80
$\rho(L) = (1 - L^2)^{d_1}\,(1 + L^2)^{d_2}$	6.24	7.59	10.19	13.86
$\rho(L) = (1 - L)^{d_1}\,(1 + L)^{d_2}\,(1 + L^2)^{d_3}$	7.08	8.60	12.02	16.97

The sample size is T = 120 and 50,000 replications wre used in each case.

TABLE 6									
Rejection frequencies of $\hat{R}$ in (11) against fractional alternatives based on the critical values obtained in Table 5. True model: $(1 - L^4)\, y_t = \varepsilon_t$, and T = 120.									
$\rho(L;\theta) = (1-L^4)^{1+\theta}$									
θ	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
	0505	0.468	0.332	0.077	**0.046**	0.442	0.915	0.994	0.999
$\rho(L;\theta) = (1-L^2)^{1+\theta_1}(1+L^2)^{1+\theta_2}$									
θ_1 / θ_2	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
-1.00	0.736	0.947	0.794	0.785	0.843	0.888	0.919	0.939	0.950
-0.75	0.960	0.627	0.754	0.764	0.888	0.949	0.973	0.984	0.989
-0.50	1.000	0.970	0.412	0.571	0.886	0.969	0.991	0.997	0.998
-0.25	1.000	1.000	0.960	0.107	0.642	0.956	0.993	0.998	0.999
0.00	1.000	1.000	0.999	0.922	**0.048**	0.725	0.980	0.997	0.999
0.25	1.000	1.000	1.000	0.999	0.857	0.204	0.763	0.985	0.999
0.50	1.000	1.000	1.000	1.000	0.997	0.814	0.717	0.778	0.987
0.75	1.000	1.000	1.000	1.000	1.000	0.996	0.812	0.961	0.784
1.00	1.000	1.000	1.000	1.000	1.000	1.000	0.993	0.822	0.995
$\rho(L;\theta) = (1-L)^{1+\theta_1}(1+L)^{1+\theta_2}(1+L^2)^{1+\theta_3}$									
(θ_1, θ_2) / θ_3	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
(-1.00, -1.00)	0.829	0.923	0.999	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, -0.50)	0.979	0.985	0.998	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.00)	0.987	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 0.50)	0.985	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-1.00, 1.00)	0.981	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -1.00)	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, -0.50)	1.000	0.959	0.533	0.815	0.998	1.000	1.000	1.000	1.000
(-0.50, 0.00)	1.000	0.994	0.945	0.985	1.000	1.000	1.000	1.000	1.000
(-0.50, 0.50)	1.000	0.994	0.996	1.000	1.000	1.000	1.000	1.000	1.000
(-0.50, 1.00)	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.00, 0.00)	1.000	1.000	0.997	0.541	**0.049**	0.687	0.999	1.000	1.000
(0.00, 0.50)	1.000	1.000	1.000	0.881	0.636	0.961	1.000	1.000	1.000
(0.00, 1.00)	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000
(0.50, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(0.50, 0.00)	1.000	1.000	1.000	1.000	0.992	0.997	1.000	1.000	1.000
(0.50, 0.50)	1.000	1.000	1.000	1.000	0.700	0.355	0.872	0.998	1.000
(0.50, 1.00)	1.000	1.000	1.000	1.000	0.817	0.452	0.904	0.997	1.000
(1.00, -1.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, -0.50)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.00)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(1.00, 0.50)	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000
(1.00, 1.00)	1.000	1.000	1.000	1.000	0.999	0.937	0.939	0.998	1.000

10,000 replications were used for each case. Sizes are in bold and the nominal size was 5%.